Some remarks on elicitible risk measures, generalized quantiles and expectiles

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Outline

- Expectiles
- Elicitability and backtesting
- Characterizations of expectiles
- Asymptotic behaviour of high expectiles
Coherent risk measures

The axiomatic approach to the problem of risk measurement originated with the seminal papers of Artzner et al. (1997, 1999) and Delbaen (2002). They introduced several natural axioms for a risk measure $\rho: L^\infty \rightarrow \mathbb{R}$:

- $\rho(X + h) = \rho(X) - h$, $\forall h \in \mathbb{R}$ (translation invariance)
- $\rho(\lambda X) = \lambda \rho(X)$, $\forall \lambda > 0$ (positive homogeneity)
- $X \leq Y$ a.s. $\Rightarrow \rho(X) \geq \rho(Y)$ (monotonicity)
- $\rho(X + Y) \leq \rho(X) + \rho(Y)$ (subadditivity).

It was pointed out that the VaR does not satisfy the subadditivity axiom and the Expected Shortfall (also known as Conditional Value at Risk or Average Value at Risk) has been suggested as an alternative.
The acceptance set $A_\rho$ of a risk measure $\rho$ is defined simply as

$$A_\rho := \{ X \in L^\infty \mid \rho(X) \leq 0 \},$$

and represents the set of positions that have a nonpositive risk. Under very weak technical assumptions, a translation invariant risk measure is determined by its acceptance set as follows:

$$\rho(X) = \inf\{ m \in \mathbb{R} \mid X + m \in A_\rho \},$$

that is the minimum amount of capital that has to be added to $X$ to make it acceptable. So, to choose a translation invariant risk measure is equivalent as specifying a notion of acceptable risk.
Acceptance sets

The properties of the risk measure \( \rho \) are fully reflected by the properties of \( A_\rho \), in particular:

- \( \rho \) is positively homogeneous if and only if \( A_\rho \) is a cone
- \( \rho \) is monotone if and only if \( X \in A_\rho \) and \( Y \geq X \Rightarrow Y \in A_\rho \)
- \( \rho \) is subadditive if and only if \( X, Y \in A_\rho \Rightarrow X + Y \in A_\rho \).

The most conservative case is \( A_\rho = L^\infty_+ \). In this case, no loss is acceptable. The corresponding risk measure is the worst case loss:

\[
\rho_A(X) = \text{ess sup}(-X)
\]

The least conservative case is \( A_\rho = \{ X \in L^\infty | E[X] \geq 0 \} \). Here, all positions with a positive expected value are acceptable. The corresponding risk measure is the expected loss:

\[
\rho_A(X) = E[-X]
\]
VaR acceptance set

In the case of $\text{VaR}_\alpha$, a position is acceptable if the probability of a loss is no greater than $\alpha$:

$$\mathcal{A}_{\text{VaR}_\alpha} = \{X \in L^\infty \mid P(X < 0) \leq \alpha\},$$

that gives

$$\rho_A(X) = -q^+_\alpha(X),$$

where $q^+_\alpha(X)$ is the right $\alpha$-quantile of $X$, defined by

$$q^+_\alpha(X) = \inf\{t \in \mathbb{R} \mid F_X(t) > \alpha\}.$$ 

Note that we may write equivalently

$$\mathcal{A}_{\text{VaR}_\alpha} = \{X \in L^\infty \text{ such that } \frac{P(X \geq 0)}{P(X < 0)} \geq \frac{1 - \alpha}{\alpha}\}.$$
Expected Shortfall acceptance set

We recall that the expected shortfall $ES_\alpha$ for a continuous distribution $X$ is given by

$$ES_\alpha(X) = -E[X|X \leq q_\alpha].$$

More generally, we have

$$ES_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha q_s(X)ds.$$

The acceptance set of the ES is given by

$$\mathcal{A}_{ES_\alpha} = \{X \in L^\infty \text{ such that } \frac{1}{\alpha} \int_0^\alpha q_s(X)ds \geq 0\}.$$  

Clearly, $\mathcal{A}_{ES_\alpha} \subset \mathcal{A}_{VaR_\alpha}$, and indeed $ES_\alpha$ is more conservative than $VaR_\alpha$. 
We now introduce $EVaR_\alpha$ (from expectile-VaR), as the monetary risk measure associated to the following acceptance set:

$$A_{EVaR_\alpha} = \{ X \in L^\infty \text{ such that } \frac{E(X^+)}{E(X^-)} \geq \frac{1 - \alpha}{\alpha} \},$$

where $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$. Note the similarity with

$$A_{VaR_\alpha} = \{ X \in L^\infty \text{ such that } \frac{P(X \geq 0)}{P(X < 0)} \geq \frac{1 - \alpha}{\alpha} \}.$$

In the case of $EVaR_\alpha$, a position is acceptable if its gain-loss ratio (Bernardo and Ledoit, 2000) also known as omega-ratio (Keating and Shadwick, 2002) is bigger than a fixed threshold.
Expectile VaR

It is immediate that $\mathcal{A}_{EVaR_{\alpha}}$ is a cone and that satisfies

$$X \in \mathcal{A}_\rho \text{ and } Y \geq X \text{ a.s. } \Rightarrow Y \in \mathcal{A}_\rho.$$ 

Moreover, since we can write

$$\mathcal{A}_{EVaR_{\alpha}} = \{X \in L^\infty \mid E[\beta X + (1 - \beta)X^+] \geq 0\}$$

with $\beta = \frac{1-\alpha}{\alpha}$, it follows that when $\alpha \leq \frac{1}{2}$

$$EVaR_{\alpha} \text{ is a coherent risk measure.}$$

Clearly $\alpha = \frac{1}{2} \Rightarrow \beta = 1 \Rightarrow EVaR_{\frac{1}{2}}(X) = E[-X]$. 
Expectiles

In the statistical literature, the expectiles $e_\alpha(X)$ has been defined by Newey and Powell (1987) as the unique solution of the equation

$$\alpha E[(X - e_\alpha(X))^+] = (1 - \alpha) E[(X - e_\alpha(X))^-],$$

for $X \in L^1$ and $\alpha \in (0, 1)$. Equivalently, this equation can be written as

$$(1 - \alpha) \int_{-\infty}^{e_\alpha} F(t)dt = \alpha \int_{e_\alpha}^{+\infty} \bar{F}(t)dt$$

or

$$\alpha = \frac{E[(X - e_\alpha)^-]}{E[|X - e_\alpha|]}.$$ 

It is easy to see that

$$E VaR_\alpha(X) = \inf\{m \mid e_\alpha(X + m) \geq 0\} = -e_\alpha(X).$$
Properties of Expectiles

We recall from Newey and Powell (1987), Jones (1994), Delbaen (2013) and Bellini et al. (2014) the main properties of expectiles. Let $X \in L^1$ and $\alpha \in (0, 1)$. Then:

- $e_\alpha$ is continuous in $\alpha$
- $e_\alpha$ is strictly increasing in $\alpha$
- if $X$ has a continuous density, then
  \[
  \frac{d e_\alpha}{d \alpha} = \frac{E[|X - e_\alpha|]}{(1 - \alpha)F(e_\alpha) + \alpha \bar{F}(e_\alpha)}
  \]
- $X \leq Y$ a.s. with $P(X < Y) > 0 \Rightarrow e_\alpha(X) < e_\alpha(Y)$
- $e_\alpha(-X) = -e_{1-\alpha}(X)$. 


Dual representation

From Delbaen (2002) we know that a coherent risk measure on $L^\infty$ has a dual representation as a worst case expected loss on a set of generalized scenarios. In the case of $EVaR_\alpha$ with $\alpha \leq \frac{1}{2}$, we get

$$EVaR_\alpha(X) = \max_{\varphi \in \mathcal{M}_{EVaR_\alpha}} E[(-X)\varphi],$$

where

$$\mathcal{M}_{EVaR_\alpha} = \left\{ \varphi \in L^\infty, \varphi \geq 0 \text{ a.s., } E[\varphi] = 1, \frac{\text{ess sup } \varphi}{\text{ess inf } \varphi} \leq \beta \right\}.$$

For the sake of comparison, in the case of $ES_\alpha$ we have a similar representation with

$$\mathcal{M}_{ES_\alpha} = \left\{ \varphi \in L^\infty, \varphi \geq 0 \text{ a.s., } E[\varphi] = 1, \varphi \leq \frac{1}{\alpha} \right\}.$$
Dual representation

Notice that $\mathcal{M}_{\text{EVaR}_\alpha}$ is convex and weakly compact in $L^1$, since for each $\varphi \in L^1$ it holds

$$\frac{1}{\beta} \leq \varphi \leq \beta.$$

This is indeed the reason why we have a 'max' and not only a 'sup' in the dual representation. An optimal scenario is given by

$$\bar{\varphi} := \frac{\alpha 1_{X > \text{EVaR}_\alpha} + (1 - \alpha) 1_{X \leq \text{EVaR}_\alpha}}{E[\alpha 1_{X > \text{EVaR}_\alpha} + (1 - \alpha) 1_{X \leq \text{EVaR}_\alpha}]}.$$

Moreover, weak compactness implies the so called Lebesgue property (continuity from below):

$$X_n \to X \text{ in } L^\infty, \|X_n\| \leq k \Rightarrow \text{EVaR}_\alpha(X_n) \to \text{EVaR}_\alpha(X).$$
A Kusuoka representation of expectiles with a one-parameter family of dyadic Kusuoka measures is the following (see also Ziegel, 2013, Pichler and Shapiro, 2013 and Delbaen, 2013):

**Theorem**

Let \( X \in L^1 \), \( 0 < \alpha \leq \frac{1}{2} \) and let \( \beta = \frac{1-\alpha}{\alpha} \). Then

\[
EVaR_\alpha(X) = \max_{\gamma \in [1,\beta]} \int CVaR(X)_s \, dm_\gamma(s),
\]

where

\[
m_\gamma := (1 - \frac{1}{\gamma}) \delta_{\frac{\gamma-1}{\beta-1}} + \frac{1}{\gamma} \delta_1.
\]

Note that \( m_\gamma(\{0\}) = 0 \) for each \( \gamma \), and that \( m_\gamma \nleq_{st} m_\gamma' \) for each \( \gamma \neq \gamma' \), that implies nonredundancy.
Robustness Properties

In robust statistics, the notion of qualitative robustness of a statistical functional corresponds essentially to the continuity with respect to weak convergence. It is generally argued that this notion is too strong for financial risk measures. Stahl et al. (2012) suggest that a more appropriate notion of robustness might be continuity with respect to the Wasserstein distance, defined as

\[ d_W(F, G) := \inf \{ E[|X - Y|] : X \sim F, \ Y \sim G \}. \]

As it is well known,

\[ d_W(F_n, F) \to 0 \iff F_n \to F \text{ weakly and } \int |x| dF_n \to \int |x| dF, \]

thus convergence in the Wasserstein distance corresponds to \( \psi \)-weak convergence with \( \psi(x) = |x| \).
We can prove that expectiles are Lipschitz with respect to the Wasserstein metric:

**Theorem**

*For all* $X, Y \in L^1$ *and all* $\alpha \in (0, 1)$ *it holds that*

$$|e_\alpha(X) - e_\alpha(Y)| \leq \beta d_W(X, Y),$$

*where* $\beta = \max\{\frac{\alpha}{1-\alpha}; \frac{1-\alpha}{\alpha}\}$.

Delbaen (2013) improved the Lipschitz constant in the case $E[X] = E[Y]$. We note that continuity of the expectiles with respect to the $\psi$-weak convergence is also a consequence of the results in Krätschmer et al. (2012); the "degree" of qualitative robustness of expectiles is the same as CVaR.
Expectiles as minimizers

Actually, Newey and Powell (1987) introduced the expectiles as the solution of the following minimization problem:

\[ e_\alpha(X) = \arg \min_{x \in \mathbb{R}} E[\alpha(X - x)^+^2 + (1 - \alpha)(X - x)^-^2]. \]

Expectiles belong to the family of generalized quantiles that are defined as

\[ x_\alpha(X) = \arg \min_{x \in \mathbb{R}} E[\alpha \Phi_1(X - x)^+ + (1 - \alpha)\Phi_2(X - x)^-], \]

with \( \Phi_1, \Phi_2 : [0, +\infty) \rightarrow [0, +\infty) \) convex, strictly increasing and satisfying \( \Phi_i(0) = 0, \Phi_i(1) = 1 \) and \( \lim_{x \rightarrow +\infty} \Phi_i(x) = +\infty \), that in turn belong to the more general class of elicitable statistical functionals.
Elicitability

Definition (informal)

A (possibly set valued) statistical functional $T : \mathcal{M}_1(\mathbb{R}) \supseteq \mathbb{R}$ is elicitable if it can be defined as the (set of) minimizers of the expected value of a suitable loss function $L : \mathbb{R}^2 \to \mathbb{R}$, that is if

$$T(F) = \arg \min_x \int L(x, y) dF(y),$$

where $\mathcal{M}_1(\mathbb{R})$ is the set of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and each $\mu \in \mathcal{M}_1(\mathbb{R})$ is identified with its d.f. $F(y) := \mu(-\infty, y]$. 
Elicitability

The economical motivation behind the definition is the following: suppose that we want to provide an incentive to a forecaster to give an accurate assessment of the quantity $T(F)$.

If the functional $T$ is elicitable and the forecaster is an expected loss minimizer, than he can be induced to report a correct forecast by means of the expected loss.

Despite this notion has its roots in decision theory (for example Savage, 1971), the term ”elicitable functional” seems to have been introduced by Osband (1985). See Gneiting (2011) and Lambert et al. (2008) and the references therein for the early history of the notion of elicitability.
Elicitability and backtesting

It has been recently suggested by many authors (see for example Gneiting, 2011, Ziegel, 2013, Bellini and Bignozzi, 2013, among many others) that elicitability might be a very important property in risk management, mainly for two reasons:

▶ it provides a natural methodology to perform backtesting
▶ it provides a natural way of comparing different procedures for computing a risk measure

See also Embrechts et al. (2013), Emmer et al. (2013) for the debate on the relative merits of elicitable risk measures with respect to other kind of risk measures.
Backtesting

For example in the case of the Value at Risk, that is defined as

$$\text{VaR}_\alpha(F) = -q^+_\alpha(F),$$

where $q^+_\alpha(F)$ is the right $\alpha$-quantile of the P/L distribution $F$, the standard way of backtesting is by counting the number $V_n$ of days in a fixed reference period of length $n$ in which a “violation” occurred:

$$V_n = \sum_{i=1}^{n} \mathbb{1}\{X_i < -\text{VaR}_\alpha, i\}.$$

Under the null hypothesis that $\text{VaR}_\alpha$ is correctly assessed, $V_n$ has a binomial $B(n, \alpha)$ distribution. This approach is however difficult to generalize to other risk measures.
Backtesting

If $T(F)$ is elicitable, a natural statistic to perform backtesting is the average realized loss

$$\hat{L} = \frac{1}{n} \sum_{i=1}^{n} L(T_i, X_i)$$

If in addition the loss function $L$ is *accuracy rewarding* in the sense that if $T(F) < x_1 < x_2$ or $x_2 < x_1 < T(F)$ then

$$\int L(x_1, y) dF(y) \leq \int L(x_2, y) dF(y),$$

then we can compare different approaches for computing $T$ (say, historical, parametric, Monte Carlo) in a natural and consistent way, by simply comparing the ex post realized expected losses.
Examples

The simplest examples of elicitable statistical functionals are the mean, the quantile interval, defined as

$$T(F) := [q^-_{\alpha}(F), q^+_{\alpha}(F)],$$

and the expectiles, corresponding to the asymmetric quadratic loss

$$L(x, y) = \alpha 1_{\{y > x\}}(y - x)^2 + (1 - \alpha) 1_{\{y < x\}}(y - x)^2.$$  

These examples belong to the class of generalized quantiles that as we saw are the minimizers of an expected loss functions of the form

$$L(x, y) = \alpha \Phi_1(y - x)^+ + (1 - \alpha) \Phi_2(y - x)^-.$$
A more restrictive definition of elicitability

In Bellini and Bignozzi (2013), we adopted the following definition of elicitability:

**Definition**

A *single valued* $T : \mathcal{M}_1(\mathbb{R}) \rightarrow \mathbb{R}$ is *elicitable* on $\mathcal{M}_T \subseteq \mathcal{M}_1$ if there exists a loss function $L(x, y)$ such that for each $F \in \mathcal{M}_T$

$$\int L(x, y) dF(y) < +\infty \text{ for each } x \in \mathbb{R} \text{ and }$$

$$T(F) = \arg \min_{x \in \mathbb{R}} \int L(x, y) dF(y), \text{ where } L \text{ satisfies}$$

a) $L(x, y) \geq 0$ and $L(x, y) = 0$ if and only if $x = y$;
b) $L(x, y)$ is increasing in the first variable for $x > y$ and decreasing for $x < y$;
c) $L(x, y)$ is continuous in $x$. 
Necessary conditions for elicitability

It is well known (Osband, 1985) that an elicitable functional has convex level sets with respect to mixtures (CxLS):

\[ T(F) = T(G) = \gamma \Rightarrow T(\lambda F + (1 - \lambda)G) = \gamma. \]

Our stronger definition enables us to prove, by means of the so-called Berge maximum theorem that provides a general sufficient condition for the continuity of an argmin, that an elicitable functional is also mixture continuous in the sense that

\[ \lambda \mapsto T(\lambda F + (1 - \lambda)G) \]

is continuous in \( \lambda \), for each \( \lambda \in [0, 1] \) and \( F, G \in \mathcal{M}_T \).
So we can summarize the following necessary conditions for elicitation:

**Theorem**

*If $T$ is elicitable on $\mathcal{M}_{1,c}(\mathbb{R})$, then:*

a) $T(F) \in [\text{ess inf}(F), \text{ess sup}(F)]$;

b) $T$ has CXLS;

c) $T$ is mixture continuous;

d) the sets $\{T \leq \gamma\}$ and $\{T \geq \gamma\}$ are convex with respect to mixtures.
Examples

Example
Define $T(F) = q_{\alpha}^{-}(F)$. Then $T$ is not elicitable according to our definition, since it is not mixture continuous, as it can be seen in simple examples. The same argument applies to $q_{\alpha}^{+}(F)$. They can however be elicited on

$$\widetilde{M} = \{ F \in \mathcal{M}_{1,c}(\mathbb{R}), \text{ } F \text{ strictly increasing} \}.$$

Example
The Expected Shortfall is not elicitable, since its level sets are not convex with respect to mixtures, as it was shown by Weber (2006) and by Gneiting (2011).
Monetary elicitable risk measures

We recall that a monetary risk measure $\rho$ is said to be a *shortfall risk measure* (Föllmer and Schied, 2002) if

$$\mathcal{A}_\rho := \{X \in L^\infty \mid u(X) \geq 0\},$$

for some nondecreasing utility function $u$.

In Bellini and Bignozzi (2013), by means of the characterization of shortfall risk measures in Weber (2006), we proved the following:

**Theorem**

*Let* $\rho : \mathcal{M}_{1,c}(\mathbb{R}) \to \mathbb{R}$ *be a monetary and elicitable risk measure with a loss function* $L(x, y)$ *that is continuous in* $y$ *and satisfies* $L(x, y) \leq \psi(y)$ *for some gauge function* $\psi$. *Then* $\rho$ *is a shortfall risk measure.*
Characterizations of expectiles

By combining this result with Weber’s (2006) characterization of coherent shortfall risk measures, we can conclude that the only coherent elicitable (under our stronger definition and additional hypotheses) risk measures are expectiles.

A similar result have been obtained by Ziegel (2013), who essentially showed, starting from the Kusuoka representation, that the only coherent risk measures with CXLS are the expectiles.

Bellini et al. (2014) showed that the expectiles are the only coherent generalized quantiles.

Although it took some time to realize,

    Expectiles are the only coherent elicitable risk measures.
Comparison between quantiles and expectiles

In the statistical literature it is usually argued that typically expectiles are closer to the centre of the distribution than the corresponding quantiles, as in the following examples:
Asymptotic properties of expectiles

Our aim is to compute the asymptotic behaviour of the expectiles $e_q(X)$ for $q \to 1$, under appropriate conditions on the right tail $\overline{F}(x) = P(X > x)$ of the distribution of the loss $X$.

This is interesting for two main reasons:

- the asymptotic expressions can be a good approximation of the true value $e_q$
- the asymptotic expressions can be used for comparison between different risk measures

Similar results in the case of $CVaR_q$ have been obtained by Hua and Joe (2011), Fang and Yang (2012) and Mao and Hu (2012) in the case of the so called Haezendonck-Goovaerts risk measures.
Asymptotic comparison between quantiles and expectiles

Let us recall the EVT distributions:

\[
\Phi_{\alpha}(x) = \begin{cases} 
0 & x \leq 0 \\
\exp(-x^{-\alpha}) & x > 0 
\end{cases}, \quad \alpha > 0
\]

\[
\Psi_{\alpha}(x) = \begin{cases} 
\exp(-(x)^{-\alpha}) & x \leq 0 \\
0 & x > 0 
\end{cases}, \quad \alpha > 0
\]

\[
\Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}.
\]

We have the following:

**Theorem**

Let \( F \in MDA(\Phi_{\alpha}) \), with \( \alpha > 1 \). If \( \alpha > 2 \), definitely \( e_q < x_q \); if \( \alpha < 2 \), definitely \( e_q > x_q \). If \( F \in MDA(\Psi_{\alpha}) \) or \( F \in MDA(\Lambda) \), then definitely \( e_q < x_q \).
Asymptotic comparison between quantiles and expectiles

Thus expectiles are always for high $q$ less conservative than the corresponding quantiles, with the only exception of paretian tails with $\alpha < 2$, that correspond to extremely heavy tails (infinite variance).

When $\alpha = 2$ several situations are possible: for a $t$ distribution with $\nu = 2$ it holds that $e_q = x_q$ for each $q \in (0, 1)$, as shown by Koenker (1993). For a Pareto distribution it is possible to compute explicitly

$$e_q = \theta \sqrt{\frac{q}{1 - q}},$$

and it holds that $e_q > x_q$ for each $q \in (0, 1)$.
First order asymptotics

We have the following first order asymptotic expressions:

**Theorem**

Let $\overline{F} \in MDA(\Phi_\alpha)$, with $\alpha > 1$. Then, for $q \to 1$,

$$e_q \sim (\alpha - 1)^{-\frac{1}{\alpha}} x_q.$$  

Let $\overline{F} \in MDA(\Psi_\alpha)$, with $\overline{F}(x) \sim C(\hat{x} - x)^\alpha$, $C > 0, \alpha > 0$; then

$$\hat{x} - e_q \sim \left[ \frac{(\hat{x} - E[X])(\alpha + 1)}{C} \right]^{\frac{1}{\alpha+1}} (1 - q)^{\frac{1}{\alpha+1}}.$$  

For comparison, when $\overline{F} \in MDA(\Phi_\alpha)$, it holds that for $q \to 1$

$$CVaR_q \sim \frac{\alpha}{\alpha - 1} x_q.$$
The Gumbel case

By means of the results of Beirlant and Teugels (1992) we can prove that for several distributions that satisfy the condition

$$- \ln(\bar{F}(x)) \sim x^\alpha L(x), \text{ for } x \to +\infty,$$

with $L(x)$ slowly varying at $+\infty$ and $\alpha > 0$, such as the Normal, the Log-normal, the Weibull and the Logistic, it holds that

$$\frac{e_q}{x_q} \to 1, \text{ for } q \to 1,$$

as in the case of ES.
Example: Student t

Expectile vs quantile
Student t with nu = 1.5

Expectile vs quantile
Student t with nu = 2.5
Example: Normal and Lognormal

Expectile vs quantile
Standard Normal distribution

Expectile vs quantile
Standard Log–Normal distribution
Empirical study: SP500 log-return

Figure: For different values of $q \in (0, 1)$ we present expectiles $e_q$ (black line) and quantiles $x_q$ (red line) for SP500 log-return data between 02/01/1957 and 14/04/2014.
Second order regular variation

Definition
A measurable function \( h : \mathbb{R}^+ \to \mathbb{R} \) is said to be of second-order regular variation with first-order parameter \( \alpha \in \mathbb{R} \) and second-order parameter \( \rho < 0 \), if there an auxiliary function \( a(t) \) with \( a(t) \to 0 \) as \( t \to \infty \) such that

\[
\lim_{t \to \infty} \frac{h(tx)}{h(t)} - x^\alpha a(t) = x^\alpha \int_{1}^{x} u^{\rho-1} \, du.
\]

Hua and Joe (2011) proved the following:

Theorem
Let \( \alpha > 0 \) and \( \rho < 0 \). Then \( F \in 2RV_{-\alpha,\rho} \) if and only if

\[
F(t) = kt^{-\alpha} \ell(t),
\]

with \( k > 0, \ell(t) \to 1 \) and \( |1 - \ell(t)| \in RV_{\rho}, \) for \( t \to +\infty \).
Second order asymptotic behaviour

When $\overline{F} \in 2RV_{-\alpha,\rho}$, Hua and Joe (2011) and Mao and Hu (2012) proved that

$$
\frac{E[(X - x)_+]}{\overline{F}(x)} \sim \frac{x}{\alpha - 1} + \frac{x a(x)}{(\alpha - 1 - \rho)(\alpha - 1)}.
$$

This the following second order expansion for $e_q$:

**Theorem**

Let $\overline{F} \in 2RV_{-\alpha,\rho}$ with $\alpha > 0$ and $\rho < 0$, i.e., $\overline{F}(t) = kt^{-\alpha}\ell(t)$. Then

$$
\frac{e_q - E[X]}{e_q^{-\alpha + 1}} \sim \frac{k}{\alpha - 1} \frac{2q - 1}{1 - q} \frac{(\alpha - 1) \ell(e_q) - \rho}{\alpha - 1 - \rho}.
$$
Examples: Pareto case

Let \( F(x) = \left( \frac{\theta}{x+\theta} \right)^\alpha \).

Then \( F(x) = \theta^\alpha x^{-\alpha} \left[ 1 - \frac{\alpha \theta}{x} + o\left( \frac{1}{x} \right) \right], \) so \( F \in 2RV_{-\alpha,-1} \).

So, we derive the following second order expression:

\[
\frac{E[(X - x)_+]}{F(x)} \sim \frac{x}{\alpha - 1} + \frac{xa(x)}{(\alpha - 1 - \rho)(\alpha - 1)} = \frac{x}{\alpha - 1} + \frac{\theta}{\alpha - 1}.
\]

Indeed, in the Pareto case, the exact mean excess function is

\[ e(x) := \frac{E[(X - x)_+]}{F(x)} = \frac{\theta + x}{\alpha - 1}, \]

Then for a Pareto the second order approximation of \( e_q \) is exact.
Examples: Pareto case

Figure: Pareto case with $\theta = 1$. The second order approximation of expectile $e_q$ for a Pareto is exactly the true value (full line); the first order approximation $e_q \sim (\alpha - 1)^{-1/\alpha} x_q$ (dashed line). Upper curves are associated to $\alpha = 1.6$, lower ones to $\alpha = 2$. 
Examples: Burr case

Let $\bar{F}(x) = (1 + x^b)^{-a}$. Then $\bar{F} \in 2RV_{-a,b,b}$.

Figure: Left: Burr case with $a = 2$ and $b = 1$ ($\alpha = 2 \rho = -1$). Right: Burr case with $a = 2$ and $b = 2$ ($\alpha = 4 \rho = -2$).
Examples: t-distribution case

In this case $F \in 2RV_{-\nu,-2}$ and for $\nu = 2$:

![Graph showing t-distribution for $\nu = 2$ (see Koenker), $\alpha = \nu$, $\rho = -2$ and $e_q = x_q$ for all $q \in (0,1)$.)]
Examples: Hall-Weiss class

If $\overline{F}(t) = \frac{1}{2} t^{-\alpha} (1 + t^\rho)$, then $E[X] = 1 + \frac{1}{2} \left( \frac{1}{\alpha - 1} + \frac{1}{\alpha - 1 - \rho} \right)$.

Figure: Left: Hall-Weiss class with $\alpha = 1.7$. Upper curves are associated to $\rho = -0.1$, lower ones to $\rho = -1$. Right: Hall-Weiss class with $\rho = -0.5$. Upper curves are associated to $\alpha = 1.7$, lower ones to $\alpha = 2$. 